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by E. F. Masur

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POST-BUCKLING STRENGTH OF REDUNDANT TRUSSES

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Synopsis

The load-carrying capacity of redundant rigid-jointed trusses after the trusses have buckled in their own plane is investigated. It is shown that (unlike similar statically determinate frame works) such trusses buckle under increasing external forces, which generally approach limiting values as buckling progresses. The computation of the ultimate load is facilitated by the use of two theorems establishing upper and lower limits of the ultimate loads. Thus, the carrying capacity of the truss can be estimated by "surrounding" the unknown load between two easily calculable values. Of these limits, the lower bound can be computed to any desired accuracy.

Introduction

The problem of the resistance of rigid-jointed trusses to buckling in their own plane has been the subject of numerous investigations. Current (1953) codes for structural design still ignore the continuity of the truss members, although few structures are constructed without some degree of rigidity. For this reason, only rigid-jointed trusses are investigated herein. However, the principles developed (in a modified and simplified form) are also applicable to pin-jointed trusses. In fact, since a buckled bar behaves similarly to a perfectly plastic one, the analysis of a buckled pin-jointed truss closely follows the analysis of an equivalent "rigid-plastic" structure.[2]

[2] "Limit Design of Beams and Frames," by H. J. Greenberg and W. Prager, *Transactions*, ASCE, Vol. 117, 1952, p. 447.

It is apparent from simple analysis that statically determinate trusses become unsafe when subjected to critical loads, that is, when subjected to external forces corresponding to neutral equilibrium. Actually such structures are incapable of supporting loads in excess of the critical loads when the structures are in the buckled state. Assuming perfect elasticity throughout the structure, the external forces remain constant during the buckling process. The constancy of the forces has been demonstrated in a series of comprehensive tests.[3]

[3] "Buckling of Rigid-Jointed Plane Trusses," by N. J. Hoff, Bruno A. Boley, S. V. Nardo, and Sara Kaufman, *Transactions*, ASCE, Vol. 116, 1951, p. 958.

It will be shown that trusses which are redundant with respect to their axial-force distribution buckle under loads that usually increase, and never decrease, as the joint rotations progress. At the same time, there occurs a redistribution of the internal forces in conjunction with the reduced chord lengths of the bent bars. In the general case, the state of stress in the truss approaches a limiting

condition corresponding to the ultimate load. The ultimate load represents the final load-bearing capacity of the structure, and is usually in excess of the loads under which the truss buckles initially.

The Equations Governing the Buckled State

In the development of the equations for the buckled state, it is assumed that the structure is subjected to a set of external loads which are known, except for a common multiplying factor λ_0. Letting ($\lambda_o X_i$, $\lambda_o Y_i$) denote the Cartesian components of the external force acting on the *i*th joint, and having S_{ij} and M_{ij} equal, respectively, the axial force and moment with which the bar connecting joints i and j acts on joint i, at which all quantities are to be measured positive as shown in Fig. 1, the equations of equilibrium for joint i are

$$\sum_j S_{ij} \cos \phi_{ij} + \lambda_o X_i = 0, \ (i = 1,2, \cdots n) \quad \text{(1a)}$$

$$\sum_j S_{ij} \sin \phi_{ij} + \lambda_o Y_i = 0, \ (i = 1,2, \cdots n) \quad \text{(1b)}$$

and

$$\sum_j M_{ij} = 0, \ (i = 1,2, \cdots r) \quad \text{(2)}$$

in which ϕ_{ij} is the inclination of the bar ij in the undeflected state, and the summation extends over all the bars connected at joint i.

If the bar ij is elastic and initially straight, and if the effect of the linear joint displacement is ignored, the moments M_{ij} can be expressed in the form:[4]

[4] "Principal Effects of Axial Load on Moment-Distribution Analysis of Rigid Structures," by B. W James, *Technical Note No. 534*, National Advisory Committee for Aeronautics, Washington, D. C., 1935.

$$M_{ij} = \frac{4 E_{ij} I_{ij}}{L_{ij}} K_{ij} (\theta_i + C_{ij} \theta_j) \quad \text{(3)}$$

in which E_{ij} is the modulus of elasticity of member ij, I_{ij} denotes the moment of inertia of bar ij, L_{ij} is the length of member ij, θ_i and θ_j represent, respectively, the angular displacement of joints i and j and are measured positive in a counterclockwise direction,

$$K_{ij} = \frac{3 \beta_{ij}}{4 \beta^2_{ij} - \alpha^2_{ij}} \quad \text{(4a)}$$

and

$$C_{ij} = \frac{\alpha_{ij}}{2 \beta_{ij}} \quad \text{(4b)}$$

in which

$$\alpha_{ij} = 6 \, \frac{1 - \epsilon_{ij} \operatorname{csch} \epsilon_{ij}}{\epsilon^2_{ij}} \quad \text{(5a)}$$

$$\beta_{ij} = 3 \, \frac{\epsilon_{ij} \coth \epsilon_{ij} - 1}{\epsilon^2_{ij}} \quad \text{(5b)}$$

and

$$\epsilon_{ij} = \sqrt{\frac{S_{ij} L^2_{ij}}{E_{ij} I_{ij}}} \quad \text{(6)}$$

It should be noted that for bars in compression ϵ becomes imaginary, in which

case the hyperbolic terms in Eqs. 5 are replaced by similar expressions involving trigonometric functions.

To ignore the effect of linear joint displacement and to establish Eq. 1 with reference to the geometry of the undeflected structure is permissible for most types of trusses. Those for which the joint translations become important, however (such as latticed struts or trusses used as lateral bracing for the upper chords of through-type bridges), are excluded.

If Eq. 3 is substituted into Eq. 2, a set of n linear, homogeneous equations involving the n unknown joint rotations, θ_i, is obtained. In order that these equations can have a nontrivial solution, it is necessary and sufficient that the determinant of the coefficients a_{ij} of the joint rotations vanish, or

$$f(\lambda_0) \equiv |a_{ij}| = 0 \qquad (7)$$

in which

$$a_{ii} = \sum_{l} \frac{4\, E_{il}\, I_{il}}{L_{il}} K_{il} \qquad (8a)$$

and

$$a_{ij} = a_{ji} = \frac{4\, E_{ij}\, I_{ij}}{L_{ij}} C_{ij}\, K_{ij}\ (i \neq j) \qquad (8b)$$

in which the summation in Eq. 8a extends over all the bars joined at joint i. Eq. 7 (the "instability equation") represents the familiar "determinant criterion."[5]

[5] "Numerical Methods for the Calculation of Elastic Instability," by B. A. Boley, *Journal of the Aeronautical Sciences*, June, 1947, p. 337.

In a statically determinate truss, the bar forces S_{ij} are uniquely determined by Eqs. 1 for any given value of λ_o. Conversely, if the truss be of the mth degree of redundancy, the internal forces can be expressed in the form:

$$S_{ij} = \sum_{\alpha=0}^{m} \lambda_\alpha S^\alpha{}_{ij} \qquad (9)$$

in which the $(m + 1)$-force systems $S^r{}_{ij}$ are subject to the restrictions

$$\sum_j S^r{}_{ij} \cos \phi_{ij} + \delta_{or} X_i = 0,\ (i = 1,2,\cdots, n) \qquad (10a)$$
$$(r = 0,1,2,\cdots, m)$$

$$\sum_j S^r{}_{ij} \sin \phi_{ij} + \delta_{or} Y_i = 0,\ (i = 1,2,\cdots n)$$
$$(r = 0,1,2,\cdots, m) \qquad (10b)$$

in which δ_{pq} is the Kronecker delta ($\delta_{pp} = 1$ and $\delta_{pq} = 0$ if $p \neq q$).

It should be noted that the force systems $S^r{}_{ij}$ are not determined uniquely by Eqs. 10. It is convenient to select the forces $S^r{}_{ij}$ $(r = 1, 2,\cdots, m)$ in such a way that

$$\sum_k \frac{S^r{}_k\, S^s{}_k\, L_k}{A_k\, E_k} = \delta_{rs}\ (r,s = 1,2,\cdots, m) \qquad (11)$$

in which S_k is the axial force in the kth-bar and the summation extends over all the bars in the truss. The instability equation (Eq. 7) for the redundant truss

takes the form:

$$f(\lambda_0, \lambda_1, \cdots, \lambda_m) = |a_{ij}| = 0 \quad \text{(12)}$$

The unknown parameters $(\lambda_0, \lambda_1, \cdots, \lambda_m)$ for the buckled truss are now determined by use of the geometric requirement that there exist a set of linear joint displacements (u_i, v_i) such that

$$\Delta L_k = (u_j - u_i) \cos \phi_{ij} + (v_j - v_i) \sin \phi_{ij} \quad \text{(13)}$$

in which ΔL_k represents the change in the chord length (of the kth-bar connecting joints i and j) from the unloaded state. If both sides of Eq. 13 are multiplied by S^r_k $(r = 1,2,\cdots, m)$ and summed over all the bars, and if Eqs. 10 and the fact that $\phi_{ij} = \phi_{ji} \pm \pi$ are considered, then (after regrouping the sums on the right side) the following set of equations is obtained:

$$\sum_k S^r_k \, \Delta L_k = 0, \; (r = 1,2,\cdots, m) \quad \text{(14)}$$

If it is possible to have a system of initial stresses,

$$S^*_k = \sum_{\alpha=1}^{m} \lambda^*_\alpha \, S^\alpha_k \quad \text{(15)}$$

then

$$\Delta L_k = \frac{(S_k - S^*_k) \, L_k}{A_k \, E_k} - \delta_k \quad \text{(16)}$$

in which δ_k represents the shortening of the chord length corresponding to the bending of the kth-bar.

It is also assumed, without loss of generality, that the state $S_{ij} = S^0_{ij}$ represents the force system in the unprestressed, unbuckled truss for λ_0 equal to unity. Since, in this case, S^*_k and δ_k both vanish for all bars, substitution of Eq. 16 into Eq. 14 leads to the set of "least-strain-energy" equations—

$$\sum_k \frac{S^0_k \, S^r_k \, L_k}{A_k \, E_k} = 0, \; (r = 1,2,\cdots, m) \quad \text{(17)}$$

In the general case, after substituting Eq. 16 into Eq. 14 and considering Eqs. 9, 11, 15, and 17, there results the system of "compatibility" equations,

$$\lambda_r = \lambda^*_r = \sum_k S^r_k \, \delta_k, \; (r = 1,2,\cdots, m) \quad \text{(18)}$$

In the Appendix it is shown that the right side of Eq. 18 can be represented by

$$\sum_k S^r_k \, \delta_k = \omega \frac{\partial f}{\partial \lambda_r}, \; (r = 0,1,2,\cdots, m) \quad \text{(19)}$$

in which ω is a measure of the magnitude of the angular displacements and is defined by

$$\omega = \frac{1}{2} \frac{\sum_i \theta^2_i}{\sum_i A_{ii}} \quad \text{(20)}$$

in which A_{ii} is the cofactor associated with a_{ii} in the instability determinant, and the summations extend over all the joints. If Eqs. 19 and 20 are substituted into Eq. 18,

$$\lambda_r - \lambda^*_r = \omega \frac{\partial f}{\partial \lambda_r}, \ (r = 1,2,\cdots, m) \quad \text{(21)}$$

The term $\partial f/\partial \lambda_r$ has been investigated[6] and can be expressed most conveniently by the equation,

[6] "The Effect of Prestressing on the Buckling Loads of Statically Redundant Rigid-Jointed Trusses," by E. F. Masur, *Proceedings*, First U. S. National Cong., of Applied Mechanics, Edwards Brothers, Inc., Ann Arbor, Mich., 1952.

$$\frac{\partial f}{\partial \lambda_r} = \sum_{i=1}^{n} \sum_{j=1}^{n} b^r_{ij} \, A_{ij}, \ (r = 1,2,\cdots, m) \quad \text{(22a)}$$

in which

$$b^r_{ii} = \frac{\partial a_{ii}}{\partial \lambda_r} = \sum_{t} \frac{E_{it} \, I_{it}}{L_{it}} K_{it} \frac{2 \, S^r_{it}}{S_{it}} (1 - 4 \, C^2_{it} \, K_{it}) \quad \text{(22b)}$$

$$b^r_{ij} = \frac{\partial a_{ij}}{\partial \lambda_r} = \frac{E_{ij} \, I_{ij}}{L_{ij}} K_{ij} \frac{2 \, S^r_{ij}}{S_{ij}} (1 + 2 \, C_{ij} - 4 \, C_{ij} \, K_{ij}), \ (i \neq j) \quad \text{(22c)}$$

and A_{ij} is the cofactor of a_{ij} in Eq. 12. The constants K and C are defined by Eqs. 4 and have been tabulated individually and in a number of combinations that are frequently of practical interest.[7]

[7] "Extended Tables of Stiffness and Carry-Over Factor for Structural Members under Axial Load," by E. E. Lundquist and W. D. Kroll, *Wartime Report L-255* (originally 4B24, 1944), National Advisory Committee for Aeronautics, Washington, D. C.

Example and General Discussion

To trace the history of the structure throughout the buckling process, the engineer is faced with the task of solving the $(m + 1)$ transcendental equations (Eqs. 12 and 21) explicitly for the $(m + 1)$ unknowns $\lambda_0, \lambda_1, \cdots, \lambda_m$ as functions of the independent variable ω. Care must be taken to admit only the smallest positive real root λ_0 since it alone corresponds to neutral equilibrium and is of interest to the engineer.

In evaluating the angular joint displacements, it should be remembered that the θ_i are proportional to the cofactors of an arbitrary row of the instability determinant. Thus, if

$$\theta_i = c_m \, A_{mi} \quad \text{(23)}$$

in which m is arbitrary, then after finding all the factors and cofactors in the determinant for a given value of ω, c_m can be computed by substituting Eq. 23 into Eq. 20, which yields

$$c^2_m = 2 \, \omega \frac{\sum_i A_{ii}}{\sum_i A^2_{mi}} \quad \text{(24a)}$$

By considering the fact that $A_{ij} = A_{ji}$, and from Eq. 12, Eq. 24a can be simplified to take the form:

$$c^2_m = \frac{2 \, \omega}{A_{mm}} \quad \text{(24b)}$$

The foregoing process was applied to the singly-redundant truss shown in Fig. 2 in which all members are assumed to be made of the same material and to have the same moment of inertia. The diagonal members are solid circular bars having twice the cross-sectional area as the nondiagonal members. The dashed lines represent the buckling mode corresponding to neutral equilibrium. The results of this study, for which the numerical computational details are available,[8] are given in Fig. 3. The ordinate represents the percentage increase of the load sustained during buckling above that load under which the unprestressed truss is about to buckle, that is, the load which is considered to be critical in conventional design. If the axial compressive unit stress in member 1–3, which corresponds to that critical load, is termed the critical stress, then the abscissa in Fig. 3 is the ratio σ of the variable maximum bending stress in bar 1–3 during buckling to the "critical stress." The selection of σ as the independent variable eliminates the influence of the diameter of bar 1–3 on the results. The history of the truss is traced for various prestressing conditions. These are identified by the parameter q^*, which represents the ratio of the initial compressive force in the nondiagonal members to the critical load.

[8] "The Stability of Statically Indeterminate, Rigid-Jointed Trusses," by E. F. Masur, thesis presented in 1951 to Illinois Inst. of Technology, at Chicago, Ill., in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mechanics.

It is apparent from Fig. 3 that with increasing σ (as buckling proceeds) the loads supported by the structure approach a limiting value, or ultimate load, which for the example shown exceeds the critical load by 88%. This ultimate load is independent of the initial stresses; in fact, it is reached even for the case of over-prestressing ($q^* > 0.793$) and for the kind of prestressing which reduces the value of the critical load ($q^* < 0$). For the case of q^* equal to 0.793, which represents the optimum initial stress system, the load remains constant; for all other cases it increases monotonically.

The results obtained for this example can be shown to hold in the general case. An inspection of Eqs. 12 and 21 shows that as ω increases (and provided that λ_r remains finite) the structure approaches a state of stress which is independent of the prestressing parameters, λ^*_r, and which is governed by the equations,

$$f = 0 \qquad (25a)$$

and

$$\frac{\partial f}{\partial \lambda_r} = 0, \; (r = 1, 2, \cdots, m) \qquad (25b)$$

The factors λ_r usually remain finite. It is possible, however, to construct trusses which exhibit no ultimate loads. Such trusses can be identified by the fact that the conditions of Theorem II (to be given subsequently) cannot be satisfied.

It has been shown[6] that Eqs. 25 are the equations to be used in determining a system of initial stresses that corresponds to a stationary value of the buckling load. It follows from Theorem I (to be introduced subsequently) that the ultimate load as determined from Eqs. 25 equals the largest of all buckling loads obtainable through prestressing.

If the initial stresses satisfy Eqs. 25 for $\omega = 0$ (to make the buckling load a maximum), Eqs. 12 and 21 are identically satisfied by λ_r being identically equal

to λ^*_r, that is constant, for all values of ω. Thus, for this system of prestresses a truss buckles similarly to a statically determinate truss with constant internal and external forces, as exemplified by the case of q^* equal to 0.793 in Fig. 3. For all other prestressing systems, the state of stress in a truss approaches that governed by Eqs. 25, whereas the value of the external loads is a monotonically increasing function of ω.[9]

[9] "Lower and Upper Bounds to the Ultimate Load of Buckled Redundant Trusses," by E. F. Masur, *Quarterly of Applied Mathematics* (publication pending).

Upper and Lower Bounds to the Ultimate Load

The example shown in Fig. 2 was selected for its relative numerical simplicity. For more intricate trusses, the calculation of the ultimate load presents formidable computational difficulties. These obstacles can be overcome by the use of two theorems which establish readily calculable lower and upper bounds to the ultimate load. Through the use of these theorems, the designer can easily determine two load values of which one is known to be smaller, the other larger, than the exact value of the ultimate load. Furthermore, the designer can approach the ultimate load "from below" as closely as he deems desirable, although a similar approach "from above" is not feasible.

For the sake of clarity it is restated that in all cases in which there is more than one applied load, the ratio between the loads is assumed to be held constant; in such cases the term "load" refers to the load parameter λ_0. In addition, a load $\lambda_o > 0$ is termed critical if there exists a set of parameters λ_r $(r = 1,2,\cdots, m)$ such that the truss is in neutral equilibrium with its internal forces identified by $(\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_m)$.

Theorem I.—The ultimate load of a redundant truss is its largest critical load. From this theorem a corollary can be deduced. If an arbitrary set of internal forces S_{ij} be in equilibrium with external loads identified by $\lambda_0 = \lambda'_0 > 0$, and if the truss be in stable equilibrium with its internal forces so defined, then

$$\lambda^U_0 > \lambda'_0 \qquad (26)$$

in which λ^U_0 identifies the ultimate load.

The usefulness of this corollary in establishing lower bounds to the ultimate load is apparent since the question of the stability of a truss for an assumed internal force system can readily be settled by a number of criteria. Among these criteria, a common one is based on the convergence or divergence of a modified moment-distribution process and was established by E. E. Lundquist[10]

[10] "Stability of Structural Members under Axial Load," by E. E. Lundquist, *Techincal Note No. 617*, National Advisory Committee for Aeronautics, Washington, D. C., 1937.

M. ASCE, and proved by N. J. Hoff.[11] For the truss shown in Fig. 2, various

[11] "Stable and Unstable Equilibrium of Plane Frameworks," by N. J. Hoff, *Journal of the Aeronautical Sciences*, March, 1941, p. 115.

arbitrary forces in bar 1–2 were assumed. Notice was then taken of the convergence or divergence of the moment-distribution procedure for different values of the applied load. The largest safe external force (found in a few trials) was

$$\lambda'_0 = \frac{30\, E\, I}{L^2} \qquad (27)$$

It is apparent that the ultimate carrying capacity of the truss in question is at least equal to this value. The correct value of the ultimate load is

$$\lambda^U{}_0 = \frac{33.7\, E\, I}{L^2} \qquad (28)$$

which exceeds the critical load of $\frac{17.90\, E\, I}{L^2}$ in the unprestressed truss by 88%.

Having thus obtained a set of lower bounds (to an arbitrary degree of accuracy) to the ultimate load, the designer can establish an upper bound by using a second theorem. This theorem is based on the kinematic consideration that a truss of m degrees of redundancy can be converted into a rigid-link mechanism of one degree of freedom by the removal of s members ($s \leq m + 1$) and by considering the remaining members to be pin-connected. If it is possible to select these S-members in such a way that the resulting mechanism has joint velocity vectors corresponding to a shortening of all the members to be removed, a second theorem can be established.

Theorem II.—If the s members be assigned compressive forces equal to $S_k = -4\,\pi^2\, E_k\, I_k\, L^{-2}{}_k$ ($k = 1,2,\cdots, s$), and if the remaining bar forces be so selected as to be in equilibrium with the forces S_k and the external loads identified by $\lambda_0 = \lambda''{}_0 > 0$, then

$$\lambda^U{}_0 \leq \lambda''{}_0 \qquad (29)$$

Although it is possible to establish an upper bound to the ultimate load, it should be recognized that for a well-designed truss the value so obtained is usually not close enough to the value of the ultimate load to be of much practical use because $\lambda''{}_0$ is computed on the basis of a collapse mode which involves no joint rotations and is therefore physically unrealistic. The usefulness of Theorem II is further restricted by the fact that the degree of approximation to the ultimate load from above is limited by the usually small number of bar selections which satisfy the stringent conditions of the theorem.

Theorem II is thus in marked contrast to the two theorems proposed by H. J. Greenberg and W. Prager,[2] M. ASCE, which established lower and upper bounds to the collapse loads of beams and bents by introducing plastic yield hinges. Fortunately, however, it is the lower bound that is of primary interest to the designer for reasons of safety. Furthermore, although the gap between the lower and upper bounds cannot be narrowed arbitrarily, the degree of accuracy obtained by following the procedure outlined in Theorem I can readily be estimated.

If the conditions of Theorem II are applied to the truss shown in Fig. 2, it is clear that an upper limit of the ultimate load can be determined by assuming a compressive force of magnitude $4\,\pi^2\, E_k\, I_k\, L^{-2}{}_k$ in bar 1–3 and in one of the nondiagonal members. This leads to

$$\lambda''{}_0 = 75.4\, \frac{E\, I}{L^2} \qquad (30)$$

This value of $\lambda''{}_0$ is much larger than the ultimate load, as was to be expected from the foregoing discussion.

Conclusions

It has been shown that current (1953) design methods, as applied to the stability of statically indeterminate trusses, leave untapped a reservoir of strength which, if utilized, can lead to more rational designs with a subsequent saving of material. It is understood, however, that this investigation is merely an introduction into a field in which little work has been done.

Before any practical benefit can be derived from this type of analysis, a number of points must be studied and clarified. Aside from the lack of experimental data, the validity of the idealizations require verification. Conversely, the effect of initial curvatures, eccentric connections, and other modifying factors on the magnitude of the ultimate load requires intensive investigation.

The assumption of elastic buckling is valid only for slender truss members. For heavier trusses, the buckling curves shown in Fig. 3 should be drastically modified for plastic yielding as buckling proceeds beyond values that depend primarily on the slenderness ratios of the bars. However, it is for such structures that prestressing may prove to be most useful.

APPENDIX

If the kth-member of a truss is subjected to a force system such as is shown in Fig. 4, the deflection y_k at any point x along the bar is governed by the differential equation,

$$\frac{d^2}{dx^2}\left(E_k\, I_k \frac{d^2 y_k}{dx^2}\right) - S_k \frac{d^2 y_k}{dx^2} = 0 \qquad (31)$$

whereas the geometric boundary conditions are given by

$$y_k|_{x=0} = y_k|_{x=L} = 0 \qquad (32a)$$

$$\left.\frac{dy_k}{dx}\right|_{x=0} = -\,\theta_i \qquad (32b)$$

and

$$\left.\frac{dy_k}{dx}\right|_{x=L} = -\,\theta_j \qquad (32c)$$

The end moments can be expressed either by Eq. 3 or by

$$M_{ij} = E_k\, I_k \left.\frac{d^2 y_k}{dx^2}\right|_{x=0} \qquad (33a)$$

$$M_{ji} = -\,E_k\, I_k \left.\frac{d^2 y_k}{dx^2}\right|_{x=L} \qquad (33b)$$

If there are two systems of bar forces S_{kp} and S_{kq}, to which there correspond two systems of deflection modes $y^p{}_k$ and $y^q{}_k$, then the term U_{pq} is defined by:

$$U_{pq} = \sum_k \int_0^L E_k\, I_k \frac{d^2 y^p{}_k}{dx^2}\frac{d^2 y^q{}_k}{dx^2}\, dx \qquad (34)$$

in which the summation extends over all the members. By partial integrations, the right side of Eq. 34 takes the form:

$$\sum_k E_k I_k \frac{d^2 y^p{}_k}{dx^2} \frac{dy^q{}_k}{dx}\bigg|_0^L - \sum_k \int_0^L \frac{d}{dx}\left(E_k I_k \frac{d^2 y^p{}_k}{dx^2}\right)\frac{dy^q{}_k}{dx}\,dx \ldots\ldots (35)$$

By considering the boundary conditions (Eqs. 32*b*, 32*c*, 33 and Eqs. 3, and through a rearrangement of the summation, the first sum in Eq. 35 can be represented by

$$\sum_i \left(\sum_j M^p{}_{ij}\right)\theta^q{}_i = \sum_i \sum_j a^p{}_{ij}\,\theta^q{}_i\,\theta^p{}_j \ldots\ldots (36)$$

in which the superscripts refer to the modes with which each term is associated. The second sum (after a partial integration and in consideration of Eq. 32*a*) is transformed into

$$\sum_k \int_0^L \frac{d^2}{dx^2}\left(E_k I_k \frac{d^2 y^p{}_k}{dx^2}\right) y^q{}_k\,dx \ldots\ldots (37)$$

If Eq. 37 (after substitution of Eq. 31) be integrated partly once more, Eq. 34 takes the form:

$$U_{pq} = \sum_i \sum_j a^p{}_{ij}\,\theta^q{}_i\,\theta^p{}_j - \sum_k S_{kp} \int_0^L \frac{dy^p{}_k}{dx}\frac{dy^q{}_k}{dx}\,dx \ldots\ldots (38)$$

Since $U_{pq} = U_{qp}$ by definition, and considering that $a_{ij} = a_{ji}$, the relationship,

$$\sum_k (S_{kp} - S_{kq}) \int_0^L \frac{dy^p{}_k}{dx}\frac{dy^q{}_k}{dx}\,dx = \sum_i \sum_j (a^p{}_{ij} - a^q{}_{ij})\,\theta^p{}_i\,\theta^q{}_j, \ldots\ldots (39)$$

is obtained.

It is of interest to note that if the force systems and deflection modes considered in Eq. 39 represent actual buckling modes the right side of the equation vanishes. If, in addition, the two force systems correspond to the same redundant parameters, λ_r $(r = 1,2,\cdots, m)$, only the load parameters, λ_0, being different, Eq. 39 reduces to the general orthogonality relationship:

$$\sum_k S^0{}_k \int_0^L \frac{dy^p{}_k}{dx}\frac{dy^q{}_k}{dx}\,dx = 0 \ldots\ldots (40)$$

if p does not equal q.

Eq. 39 can also be utilized to develop Eqs. 19. In fact, since the force systems considered are entirely arbitrary, it is permissible to select them in such a way that

$$S_{kp} - S_{kq} = (\lambda^p{}_r - \lambda^q{}_r)\,S^r{}_k \ldots\ldots (41)$$

in which r is an arbitrary, nonnegative integer not exceeding m. That is, the two force systems are chosen in accordance with Eq. 9, subject to the condition that all parameters except one (λ_r) be the same for both. In this case Eq. 39 (on consideration of Eq. 41) takes the form:

$$\sum_k S^r{}_k \int_0^L \frac{dy^p{}_k}{dx}\frac{dy^q{}_k}{dx}\,dx = \sum_i \sum_j \frac{a^p{}_{ij} - a^q{}_{ij}}{\lambda^p{}_r - \lambda^q{}_r}\,\theta^p{}_i\,\theta^q{}_j \quad \text{(42)}$$

If the two force systems and deflection modes are permitted to approach one another, and as the limit of $(\lambda^p{}_r - \lambda^q{}_r)$ approaches zero, Eq. 42 is transformed into the relationship:

$$\frac{1}{2}\sum_k S^r{}_k \int_0^L \left(\frac{dy_k}{dx}\right)^2 dx = \frac{1}{2}\sum_i \sum_j \frac{\partial a_{ij}}{\partial \lambda_r}\,\theta_i\,\theta_j,\ (r = 0, 1, 2, \cdots, m) \quad \text{(43)}$$

Eq. 43 can also be derived in a direct manner.[8]

It is apparent that the left side of Eq. 43 coincides with that of Eq. 19 since

$$\delta = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx \quad \text{(44)}$$

is the customary first approximation to the difference between the chord length and the arc length of a bent bar.

For the right side of Eq. 43, if the angles θ_i satisfy (nontrivially) the set of equations,

$$\sum_{j=1}^{n} a_{ij}\,\theta_j = \mu\,\theta_i\ (i = 1, 2, \cdots, n) \quad \text{(45a)}$$

then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,\theta_i\,\theta_j = \mu \sum_{i=1}^{n} \theta^2{}_i \quad \text{(45b)}$$

It is also assumed that all the terms appearing in Eqs. 45*a* are differentiable functions of the parameter λ.

If both sides of Eq. 45*b* are differentiated with respect to λ, and in consideration of Eqs. 45*a* and of $a_{ij} = a_{ji}$, the equation,

$$\sum_i \sum_j a'_{ij}\,\theta_i\,\theta_j = \mu' \sum_i \theta^2{}_i \quad \text{(46)}$$

is obtained in which, for simplicity, primes have been used to denote derivatives with respect to λ. However, a necessary and sufficient condition for the nontriviality of the solution of Eq. 45*a* is given by

$$g(\lambda) = f[\mu(\lambda), \lambda] = |b_{ij}| \equiv |a_{ij} - \mu\,\delta_{ij}| = 0 \quad \text{(47)}$$

Therefore, it follows that

$$g' = \frac{\partial f}{\partial \mu}\mu' + \frac{\partial f}{\partial \lambda} = -\mu' \sum_i B_{ii} + \frac{\partial f}{\partial \lambda} = 0 \quad \text{(48a)}$$

or

$$\mu' = \frac{\dfrac{\partial f}{\partial \lambda}}{\sum_i B_{ii}} \quad \text{(48b)}$$

in which B_{ii} is the cofactor of b_{ii} in Eq. 47.

If the angles θ_i correspond to a buckling mode (that is, to the case of $\mu = 0$) the determinant appearing in Eq. 47 is the same as that in the instability equation (Eq. 12) and $B_{ii} = A_{ii}$. Thus, by substituting Eqs. 44 and 46 into Eq. 43, and in consideration of Eq. 48*b*, the relationship,

$$\sum_k S^r_k \, \delta_k = \omega \frac{\partial f}{\partial \lambda_r} \; (r = 0,1,2,\cdots, m) \quad \text{(19)}$$

in which

$$\omega = \frac{1}{2} \frac{\sum_i \theta^2_i}{\sum_i A_{ii}} \quad \text{(20)}$$

is determined.

ACKNOWLEDGMENT

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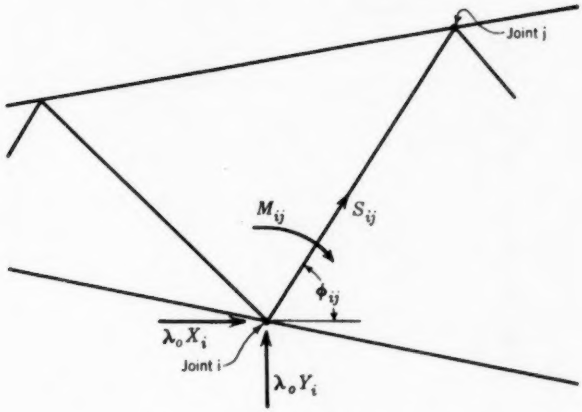

FIG. 1.—DIAGRAM OF A TYPICAL JOINT

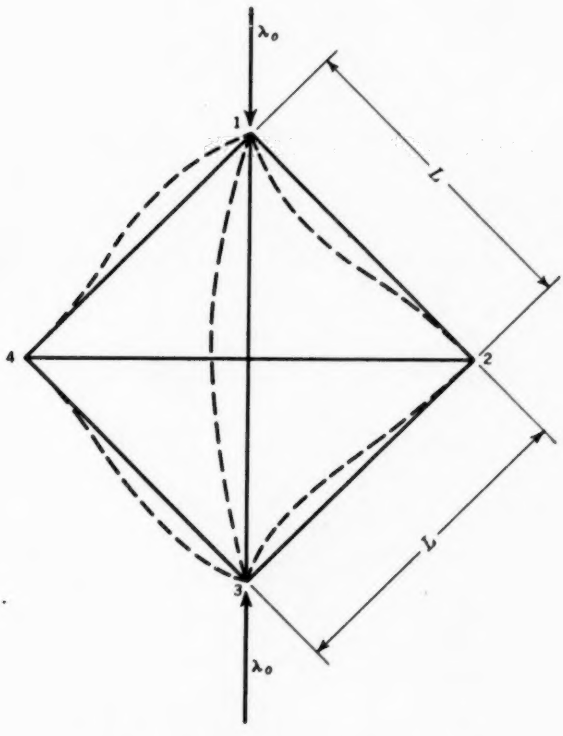

FIG. 2.—EXAMPLE OF A SINGLY REDUNDANT TRUSS

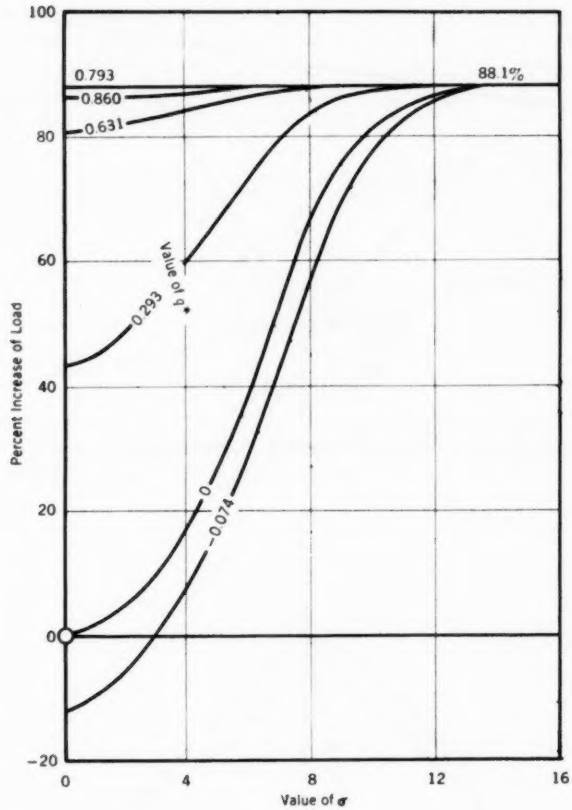

Fig. 3.—The Percentage Increase of the External Load in the Buckled State

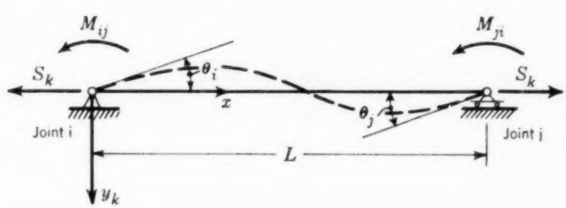

Fig. 4.—Diagram of a Typical Bar